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ON A VARIATIONAL PROBLEM WITH UNKNOWN BOUNDARIES AND THE DETERMINATION OF OPTIMAL SHAPES OF ELASTIC BODIES

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The optimization problem is considered for a partial differential equation of elliptic type. The boundary of the domain in which the equation is given emerges as the control function and is to be determined from the condition of the extremum of the integral of the solution of the boundary value problem. Seeking the extremals is reduced to solving a variational problem without differential constraints. Necessary conditions for optimality are obtained, and shapes of elastic bars possessing the maximum stiffness under torsion are found with their aid.

1. Formulation of the optimization problem and elimination of the differential constraint. We consider a boundary value problem for the partial differential equation

$$(a\varphi_x - c\varphi_y)_x + (b\varphi_y - c\varphi_x)_y + m = 0 \quad (x, y) \in D \tag{1.1}$$

$$\varphi = 0 \quad (x, y) \in \Gamma \tag{1.2}$$

The coefficients a, b, c of (1.1) are assumed given functions of the variables x, y, and $m > 0$ is a given constant, Γ is the boundary of a simply connected domain D.

Let us formulate the following optimization problem. Determine the smooth closed line Γ satisfying the isoperimetric condition of the constant area of the domain D

$$\iint_D dx\, dy = S \tag{1.3}$$

and such that a maximum of the integral functional

$$K(\Gamma) = \iint\limits_D \varphi(x, y)\, dx\, dy \to \max_\Gamma \tag{1.4}$$

is obtained for the solution $\varphi(x, y)$ of the boundary value problem (1.1),(1.2) with a given boundary Γ. The quantity S in (1.3) is a specified constant.

The optimization problem (1.1)–(1.4) refers to isoperimetric variational problems with differential constraints. The role of the differential constraint, of the isoperimetric condition, and of the control function is played, respectively, by (1.1), (1.3), and the shape of the contour Γ.

We shall henceforth assume that the coefficients of (1.1) satisfy the conditions

$$a > 0, \quad ab - c^2 > 0 \tag{1.5}$$

The assumption (1.5) permits reduction of the optimization problem (1.1)–(1.4) to a variational problem without differential constraints. In fact, for a given contour Γ and upon compliance with condition (1.5) the solution of the boundary value problem (1.1), (1.2) determines a minimum of the functional

$$J = \iint\limits_D (a\varphi_x^2 - 2c\varphi_x\varphi_y + b\varphi_y^2 - 2m\varphi)\, dx\, dy \tag{1.6}$$

considered in the class of functions $\varphi = \varphi(x, y)$, satisfying the boundary condition (1.2). It is easy to see the validity of this assertion by writing the Euler equation for the functional (1.6) and noting that it agrees with (1.1), as well as noting that the known sufficient conditions for the absolute minimum of a quadratic functional (see [1], say) agree in the case under consideration with the inequalities (1.5).

To accomplish the reduction to a problem without differential constraints, let us show that

$$J = -mK \tag{1.7}$$

holds for the function $\varphi(x, y)$, minimizing the functional (1.6) under the condition (1.2). The equality (1.7) is a direct extension of the relationship for the Poisson equation [2] to (1.1). It results from the following transformation in which (1.1), (1.2), (1.4), (1.6) are used:

$$mK = \iint\limits_D \varphi m\, dx\, dy = -\iint\limits_D \varphi\,[(a\varphi_x - c\varphi_y)_x + (b\varphi_y - c\varphi_x)_y]\, dx\, dy =$$

$$\iint\limits_D [\varphi_x(a\varphi_x - c\varphi_y) + \varphi_y(b\varphi_y - c\varphi_x)]\, dx\, dy -$$

$$\int\limits_\Gamma \varphi\,((a\varphi_x - c\varphi_y)\, dy - (b\varphi_y - c\varphi_x))\, dx =$$

$$\iint\limits_D (a\varphi_x^2 - 2c\varphi_x\varphi_y + b\varphi_y^2)\, dx\, dy = J + 2mK$$

The relationship (1.7) can be written as follows:

$$K = -\frac{1}{m}\min_\varphi J$$

Hence, and from (1.5), (1.6) we finally obtain

$$K_* = \max_\Gamma K(\Gamma) = -\min_\Gamma \min_\varphi \frac{1}{m} \iint\limits_D (a\varphi_x^2 - 2c\varphi_x\varphi_y + b\varphi_y^2 + 2m\varphi)\, dx\, dy \tag{1.8}$$

Therefore, the initial optimization problem (1.1)–(1.4) has been reduced to the variational problem (1.2), (1.3), (1.8). To solve the problem (1.2), (1.3), (1.8), the minimum with respect to φ and Γ should be calculated successively. The inner minimum with respect to φ in (1.8) is calculated for a given shape of the contour Γ and the boundary condition (1.2). The outer minimum with respect to Γ is sought under the isoperimetric condition (1.3).

2. The necessary condition for an extremum. We consider the variational problem (1.2), (1.3), (1.8) and derive the conditions satisfied by the extremals (the contour Γ and the function $\varphi(x, y)$). To this end, we write the expression for the first variation of the integral (1.8)

$$\delta J = -2\iint \delta\varphi\,[(a\varphi_x - c\varphi_y)_x + (b\varphi_y - c\varphi_x)_y + m]\,dx\,dy + \qquad (2.1)$$
$$2\int_\Gamma \delta\varphi\,[(a\varphi_x - c\varphi_y)\,dy - (b\varphi_y - c\varphi_x)\,dx] +$$
$$\int_\Gamma \delta f\,(a\varphi_x^2 + b\varphi_y^2 - 2c\varphi_x\varphi_y - 2m\varphi)\,ds$$

Here the δf denotes the normal displacement of boundary points. Since the condition (1.2) holds on the contour Γ, then $\delta\varphi = 0$ $((x, y) \in \Gamma)$, and therefore, the second integral in the right side of (2.1) is zero. Moreover, the term $2m\varphi$ in the third integral vanishes as a consequence of condition (1.2). Furthermore, using the arbitrariness of the function $\delta\varphi$ as well as the extremum condition $\delta J = 0$ and standard reasoning [3], we obtain that the function $\varphi(x, y)$ which determines the extremum of the functional under consideration makes the expression in the square brackets in the first integral of (2.1) zero. The Euler equation in φ agrees with (1.1). We henceforth assume that the function φ satisfies this equation. Consequently for the first variation to vanish $(\delta J = 0)$ it is necessary that the equality

$$\int_\Gamma \delta f\,(a\varphi_x^2 - 2c\varphi_x\varphi_y + b\varphi_y^2)\,ds = 0 \qquad (2.2)$$

holds.

According to (1.3), the variation δf should satisfy the following isoperimetric condition:

$$\int_\Gamma \delta f\,ds = 0 \qquad (2.3)$$

From (2.2) and (2.3) we have

$$a\varphi_x^2 - 2c\varphi_x\varphi_y + b\varphi_y^2 = \lambda^2, \quad (x, y) \in \Gamma \qquad (2.4)$$

where λ^2 denotes an unknown constant (Lagrange multiplier). Condition (2.4) determines the optimal contour and closes the boundary value problem (1.1), (1.2) together with (1.3) (the value of the constant λ^2 is determined from (1.3)).

3. Optimal shape of a twisted elastic bar. We consider the problem of torsion of an elastic anisotropic bar in a rectangular xyz coordinate system. Let the z-axis be parallel to the bar axis. Symbol D denotes the domain of a transverse section through the bar by the xy plane, Γ is the boundary of the domain D. We assume the bar to be continuous and, therefore, the domain D is simply connected. Let the torsion occur around the z-axis. We express the nonzero components of the stress tensor τ_{xz} and τ_{yz} in terms of the stress function $\varphi(x, y)$ using the following relationships:

$$\tau_{xz} = \theta\varphi_y, \quad \tau_{yz} = -\theta\varphi_x \qquad (3.1)$$

where θ is the angle of twist per unit length of the bar. The stress function $\varphi(x, y)$ is determined (see [4]) from the solution of the boundary value problem for the equation

$$a\varphi_{xx} - 2c\varphi_{xy} + b\varphi_{yy} = -2 \tag{3.2}$$

with boundary condition (1.2). The a, b, c in (3.2) denote the strain coefficients of the anisotropic material. These coefficients satisfy conditions (1.5) (see [4]). Let us assume the bar length and volume given. This assumption results in the isoperimetric condition (1.3).

We pose the problem of seeking the bar shape (the shape of its cross section D) for which condition (1.3) is satisfied and the maximum bar stiffness under torsion is achieved

$$K = 2\iint_D \varphi \, dx \, dy \to \max_\Gamma \tag{3.3}$$

The torque M, the stiffness K and the angle of twist θ per unit length of the bar are connected by the relationship $M = K\theta$.

The optimization problem (3.2), (3.3) formulated is a particular problem relative to (1.1) — (1.4), and hence, condition (2.4) can be used to determine the optimal shape of the bar. The solution of the boundary value problem (1.2), (2.4), (3.2) with condition (1.3) is

$$\varphi = \frac{1}{2(ab - c^2)}\left[\frac{S}{\pi}\sqrt{ab - c^2} - bx^2 - 2cxy - ay^2\right] \tag{3.4}$$

$$\Gamma\colon\ bx^2 + 2cxy + ay^2 = \pi^{-1}S\sqrt{ab - c^2} \tag{3.5}$$

The stiffness of a bar of the optimal section (3.6), calculated on the basis of (3.3) — (3.5) is

$$K_* = \frac{S^2}{2\pi\sqrt{ab - c^2}} \tag{3.6}$$

Let us compare the stiffnesses of bars of optimal cross section (3.5) and circular cross section. Taking into account that the stiffness of a bar of circular section is $K_0 = S^2 / \pi(a + b)$ and assuming the cross-section areas of the bars to be equal, we arrive at the following formula to estimate the gain due to optimization

$$\frac{K_* - K_0}{K_0} = \frac{1}{\beta} - 1, \quad \beta = \frac{2\sqrt{ab - c^2}}{a + b}$$

The deformation coefficients a, b, c of an anisotropic material satisfy the inequalities (1.5). Taking this into account, it is easy to show that $0 \leqslant \beta \leqslant 1$. For an orthotropic material $c = 0$, $a = 1/G_1$, $b = 1/G_2$, where G_1 and G_2 denote the shear moduli corresponding to the x and y coordinates. In this case, the efficiency of the optimization is estimated by the formula $(K_* - K_0) / K_0 = (G_1 + G_2) / 2\sqrt{G_1G_2} - 1$. It is seen from this relationship that the gain due to optimization increases both as $G_1 / G_2 \to 0$ and as $G_1 / G_2 \to \infty$, i.e. the relative gain increases as the degree of anisotropy increases. The minimum gain, zero, is obtained for $G_1 = G_2 = G$, i.e. for an isotropic material. In this case $K_* = K_0 = GS^2 / 2\pi$, and the optimal section is a circle.

Optimality of a circular section for a bar of homogeneous isotropic material has been proved on the basis of the theorem on symmetrization in [5].

We note that the optimality condition (2.4) for a bar from isotropic homogeneous material means constancy of the tangential stress on the contour Γ. In fact on the basis of (3.1) and (2.4) ($c = 0$, $a = b = 1 / G = \text{const}$), we have

$$\tau^2 = \tau_{xz}^2 + \tau_{yz}^2 = \theta^2(\varphi_y^2 + \varphi_x^2) = \text{const} \tag{3.7}$$

If only a part of the contour Γ_1 is varied while the rest Γ_2 is kept fixed, the form of the optimality condition on the required curve remains as before.

Note that condition (2.4), or equivalently, condition (3.7) can be given a more universal form in the case of a homogeneous isotropic material and this condition is used below in this form. To do this, let us pass from differentiation with respect to the coordinates x, y to evaluation of the derivatives with respect to the tangential and normal directions to the contour Γ. Taking account of condition (1.2), we arrive at the following relation:

$$(\partial\varphi/\partial n)^2 = \lambda^2, \quad (x, y) \in \Gamma \tag{3.8}$$

where λ^2 is a known constant.

Note 1. Determination of the shape of an elastic bar possessing minimum cross-sectional area, or equivalently, minimum weight for a given stiffness, reduces to a dual problem. Carrying out an analysis completely analogous to that presented in Sect. 1 permits elimination of the differential constraint and reduction of seeking the boundary of the cross-sectional domain to the solution of the following isoperimetric variational problem:

$$K = -\frac{1}{2}\min_\varphi J = K', \quad S = \iint_D dx\, dy \to \min_\Gamma$$

where K' is a given constant. The solution of this problem is found from (3.4) — (3.6) by a simple computation. We have

$$\Gamma: bx^2 + 2cxy + ay^2 = (ab - c^2)^{3/4}\sqrt{2K'/\pi}, \quad S = \sqrt{2\pi K'}\,(ab - c^2)^{1/4}$$

for the optimal contour and the minimizing functional.

Note 2. Maximizing the stiffness of a bar of inhomogeneous elastic material is possible also because of the optimal distribution of inhomogeneities over the section. Paper [6] is devoted to this question. Another problem of optimizing an inhomogeneous bar was solved in [7], where the optimal mutual disposition of materials over a section was sought under the assumption that the bar is composed of two materials with different plasticity constants. All the analyses in [7] were conducted within the framework of the theory of ultimate plastic design.

4. Stiffness optimization problems in the case of a not simply connected section. We consider the problem of torsion of a homogeneous isotropic prismatic bar with a section not simply connected. For brevity in the exposition, we consider the cross-sectional domain D to be doubly-connected. Symbols Γ_0 and Γ denote the inner and outer boundary of the domain D, respectively. For given boundaries Γ_0 and Γ the torsion problem reduces to seeking a stress function $\varphi(x, y)$ from the solution of the boundary value problem [2, 8]

$$\varphi_{xx} + \varphi_{yy} = -2 \tag{4.1}$$

$$\varphi = 0, \quad (x, y) \in \Gamma; \quad \varphi = C, \quad (x, y) \in \Gamma_0 \tag{4.2}$$

$$\int_{\Gamma_0} \frac{\partial\varphi}{\partial n}\, ds = 2\Omega \tag{4.3}$$

where Ω is the area of the domain bounded by the contour Γ_0. The constant C in (4.2) is an unknown quantity and condition (4.3) is used to determine it. The following expressions hold for the stresses τ_{xz}, τ_{yz} and the torsional stiffness K:

$$\tau_{xz} = G\Theta\varphi_y, \quad \tau_{yz} = -G\Theta\varphi_x \tag{4.4}$$

$$K = 2\left(\iint_D \varphi\, dx\, dy + C\Omega\right) \qquad (4.5)$$

The function φ connected with the stress tensor components by means of the relationships (4.4), differs by a constant factor G (the shear modulus) from the stress function introduced in the preceding section using (3.1). The case of a homogeneous isotropic material, when $c = 0, G_1 = G_2 = G$, is kept in mind. In particular, this explains the stiffness K defined according to (4.5) having a dimensionality of a fourth power in the length and differing from the stiffness K' in Sect. 3 by the factor G $(K' = GK)$.

The contour Γ_0, and the area Ω of the domain bounded by this contour are thereby assumed given. The area of the domain D bounded by the contours Γ_0 and Γ is also considered given. The boundary Γ is fixed beforehand and is to be determined.

The optimization problem consists of seeking the shape of the contour Γ which will maximize the functional (4.5) under the conditions (1.3),(4.1) — (4.3).

It can be shown that the condition to determine the optimal contour Γ remains as before and agrees with (3.8) in the case of the doubly-connected domain under consideration.

To obtain this condition, let us use a method which can be applied even in the case of an arbitrary n-connected domain. We introduce the torsion function ψ connected with the stress function φ by the relationships $\psi_x = \varphi_y + y$, $\psi_y = -\varphi_x - x$. We have the following Neumann problem for the function so introduced:

$$\psi_{xx} + \psi_{yy} = 0, \quad (x, y) \in D; \quad \frac{\partial \psi}{\partial n} = yn_x - xn_y, \ (x, y) \in \Gamma_0 + \Gamma \qquad (4.6)$$

where n_x, n_y are projections of the unit normal to the boundary of the domain D on the x and y axes. The bar stiffness K is computed in terms of ψ from the formula

$$K = I + \iint_D (x\psi_y - y\psi_x)\, dx\, dy, \quad I = \iint_D (x^2 + y^2)\, dx\, dy \qquad (4.7)$$

The torsion function can be found for given boundaries of the domain D by solving the variational problem [2]

$$J_1 = \frac{1}{2}\iint_D [(\psi_x - y)^2 + (\psi_y + x)^2]\, dx\, dy \to \min_\psi \qquad (4.8)$$

We note that it is not here required that the comparison functions ψ should satisfy the boundary condition (4.7) since this condition is "natural" for the functional (4.8). We have $J_1 = {}^1/_2 K$ (see [2]) for the function ψ achieving the minimum of the integral (4.8) (and not for an arbitrary comparison function). This relationship permits writing the optimization problem as follows:

$$K_* = \max_\Gamma \min_\psi 2J_1 \qquad (4.9)$$

The maximum with respect to Γ in (4.9) is computed under the isoperimetric condition (1.3). Furthermore, writing down the expression for the first variation in the integral J_1 and noting that the constraint $\int \delta f ds = 0$ holds because of (1.3) (the integral is taken over the contour Γ), we obtain an equation and boundary conditions (4.6) as the necessary conditions for the extremum as well as a condition of the optimality of the contour Γ

$$(\psi_x - y)^2 + (\psi_y + x)^2 = \text{const} \qquad (4.10)$$

Returning to the stress function φ in (4.10), we arrive at the condition $\varphi_x^2 + \varphi_y^2 = \lambda^2$, which can be written in the form of (3.8).

Let us study some properties of the optimal solutions. Applying the Bredt theorem to the contour Γ and using the optimality condition (3.8), we obtain an expression relating the valne of the constant λ from (3.8) to the length l of the optimal contour and the area $\Omega + S$

$$2(\Omega + S) = -\int_\Gamma \frac{\partial\varphi}{\partial n} ds = \lambda l \tag{4.11}$$

In particular, $\lambda \neq 0$ and $\lambda \neq \infty$ follow from (4.11).

We show that the optimal contour is smooth and has neither projecting nor entering angles. Reasoning the converse, let us first assume the presence of a projecting angle on the optimal contour. Then approaching the apex of this angle along the contour Γ we will have $\tau = \partial\varphi / \partial n \to 0$ (see [2], for example), and therefore the condition $\lambda \neq 0$ is violated in this case. Now, we assume that there is an entering angle on the contour Γ Then approaching the vertex of this angle along Γ, the quantity $\tau = \partial\varphi / \partial n$ tends to infinity, and therefore, the assumption $\lambda \neq \infty$ is violated.

5. Determination of the optimal shape by the small parameter method. We obtain the solution of the problem examined in Sect. 4 in the case of a thin-walled bar. For convenience, the new coordinate system st related to the reference line Γ_0 is introduced. The coordinate s of the point $P \in D$ is measured along Γ_0 from some point $O \in \Gamma_0$ to the intersection A of Γ_0 with the normal (to Γ_0) passing through the point P. The coordinate t equals the length of the segment AP. Let $\rho = \rho(s)$ and $h = h(s)$ denote the radius of curvature of the reference line Γ_0 and the equation of the contour Γ.

The assumption about the bar being thin-walled means that (L is the length of the contour Γ_0)

$$\max{}_s h(s) = H \ll L \quad (0 \leqslant s \leqslant L)$$

i.e. the ratio $H / L = \varepsilon$ is a small number ($\varepsilon \ll 1$).

Let us assume that the contour Γ_0 has no strongly curved sections, i.e. that

$$\min_s \rho(s) \sim L \tag{5.1}$$

We write the main relationships (1.3), (3.8), (4.1) — (4.3) and (4.5) in the new coordinate system and pass to new variables and notation

$$s = Ls', \quad t = Ht', \quad h = Hh', \quad \varphi = HL\varphi', \quad \Omega = L^2\Omega' \tag{5.2}$$

$$S = HLS', \rho = L\rho', \quad K = HL^3K', \quad C = HLC', \quad \lambda = L\lambda'$$

(we henceforth omit the primes). We obtain

$$(T\varphi_t)_t + \varepsilon^2(T^{-1}\varphi_s)_s = -2\varepsilon T, \quad T = 1 + \frac{\varepsilon t}{\rho} \tag{5.3}$$

$$\varphi_t(s, h) = -\lambda, \quad \varphi(s, 0) = C, \quad \varphi(s, h) = 0$$

$$\int_0^1 \varphi_t(s, 0)\, ds = -2\Omega, \quad \int_0^1 \left(h + \frac{\varepsilon h^2}{2\rho}\right) ds = S$$

$$K = 2\left(C\Omega + \varepsilon \int_0^1\int_0^h T\varphi\, dt\, ds\right)$$

To solve this problem, we apply the small parameter method and we seek the functions φ, h and the unknown constants C, λ and K as series in the parameter ε

$$\varphi = \varphi^\circ + \varepsilon\varphi^1 + \varepsilon^2\varphi^2 + \ldots \tag{5.4}$$

Analogous expansions are used for h, C, λ and K.

We write down the equations to determine the zero, first and second order approximations. To this end we substitute the representation (5.4) into the relationships (5.3) and equate terms with identical powers of ε. We consequently arrive at boundary value problems whose successive solution permits the determination of all the quantities required. To determine the zero order approximations, we have the following boundary value problem

$$\varphi_{tt}^\circ = 0, \ \varphi^\circ(s, 0) = C^\circ, \quad \varphi^\circ(s, h^\circ) = 0 \tag{5.5}$$

$$\varphi_t^\circ(s, h^\circ) = -\lambda^\circ \tag{5.6}$$

$$\int_0^1 \varphi_t^\circ(s, 0)\, ds = -2\Omega, \quad \int_0^1 h^\circ\, ds = S \tag{5.7}$$

Taking account of the properties of the required zero approximation functions, we write the boundary value problem for the first approximation

$$\varphi_{tt}^1 = -2 - \rho^{-1}\varphi_t^\circ, \quad \varphi^1(s, 0) = C^1, \quad \varphi^1(s, h^\circ) = 0 \tag{5.8}$$

$$\varphi_t^1(s, h^\circ) = -\lambda^1 \tag{5.9}$$

$$\int_0^1 \varphi_t^1(s, 0)\, ds = 0, \quad \int_0^1 h^1\, ds = -\frac{1}{2}\int_0^1 \frac{(h^\circ)^2}{\rho}\, ds \tag{5.10}$$

The boundary value problem for the second order approximation, taking (5.5)—(5.10) into account, has the form

$$\varphi_{tt}^2 = -\rho^{-1}[(t\varphi_t^1)_t + 2t] - \varphi_{ss}^\circ, \quad \varphi^2(s, 0) = C^2 \tag{5.11}$$

$$\varphi^2(s, h^\circ) = \lambda^1 h^1 + \lambda^\circ h^2$$

$$\varphi_t^2(s, h^\circ) = (2 + \rho^{-1}\lambda^\circ)\, h^1 - \lambda^2$$

$$\int_0^1 \varphi_t^2(s, 0)\, ds = 0, \quad \int_0^1 h^2\, ds = -\int_0^1 \frac{h^\circ h^1}{\rho}\, ds$$

Using the appropriate relationships from (5.3), (5.4) and (5.5), we obtain the following expression for the stiffness:

$$K = K^\circ + \varepsilon K^1 + \varepsilon^2 K^2 + \ldots = 2C^\circ\Omega + 2\varepsilon\left(\int_0^1\int_0^{h^\circ} \varphi^\circ\, dt\, ds + C^1\Omega\right) + \tag{5.12}$$

$$2\varepsilon^2\left(\int_0^1\int_0^{h^\circ} \varphi^1\, dt\, ds + C^2\Omega\right) + O(\varepsilon^3)$$

We find the solution of the problem in the zero approximation. It follows from the equation and boundary conditions (5.5) that $\varphi^\circ = C^\circ(1 - t/h^\circ)$. The expression for φ° and the optimality condition (5.6) yield $h^\circ = C^\circ(\lambda^\circ)^{-1}$. Substituting the functions φ° and h° found into the isoperimetric condition (5.7), we find the constants λ° and C°. We finally have

$$h^\circ = S, \quad \varphi^\circ = 2S\Omega\left(1 - \frac{t}{S}\right), \quad K^\circ = 4S\Omega^2, \quad \lambda^\circ = 2\Omega, \quad C^\circ = 2S\Omega \tag{5.13}$$

Thus, in the zero approximation the optimal thickness distribution is constant in the case of slight curvature of the contour Γ_0 (the assumption (5. 1)).

Let us determine the first approximations. To do this, we integrate (5. 8) and determine the integration constants from the boundary conditions (5. 8), while the function h^1 and the constants λ^1 and C^1 from the relationships (5. 9) and (5. 10). We consequently obtain the following expressions for the required first approximation:

$$h^1 = -2\frac{S^2}{\rho}, \quad \varphi^1 = t^2\left(\frac{\Omega}{\rho} - 1\right) + 2S\Omega\left(\Pi_1 - \frac{1}{\rho}\right)t + S^2 - 2\Omega S^2\Pi_1 \tag{5.14}$$

$$C^1 = S^2(1 - 2\Omega\Pi_1), \quad \lambda^1 = 2S(1 - \Omega\Pi_1), \quad K^1 = 4\Omega S^2(1 - \Omega\Pi_1)$$

$$\Pi_1 = \int_0^1 \frac{ds}{\rho}$$

The influence of the curvature of the inner contour Γ_0 on the optimal shape of the outer boundary Γ is taken into account in formula (5. 14) for h^1. The expression for the optimal thickness distribution of the bar

$$h = h^\circ + \varepsilon h^1 = S\,(1 - \varepsilon\, S/2\rho)$$

(in the dimensional variables $h = SL^{-1}\,(1 - S\,/\,2\rho L)$) shows that the thickness h decreases as the curvature for appropriate points of the contour Γ_0 increases.

Analogously, by solving the boundary value problem (5. 11), all the required second approximation quantities are determined. We present the expressions found here for h^2 and the constant C^2

$$h^2 = -\frac{S^3}{2\rho^2}\left(3 + \frac{\rho}{\Omega} - 4\rho^2\Pi_2 - \frac{\rho^2}{\Omega}\Pi_1\right) \tag{5.15}$$

$$C^2 = \frac{2S^3}{3}(\Omega\Pi_2 - 2\Pi_1) + 2S^3\Omega\Pi_1^2, \quad \Pi_2 = \int_0^1 \frac{ds}{\rho^2}$$

The correction K^2 is determined by using the constant C^2 and the appropriate zero and first approximations in (5. 12).

Using the expressions found and passing to the original dimensional quantities (5. 2), we obtain the following formula for the stiffness of an optimal bar (in dimensional variables):

$$K = \frac{4S\Omega^2}{L^2} + \frac{4S^2\Omega}{L^2}\left(1 - \frac{\Omega}{L^2}\int_0^L \frac{ds}{\rho}\right) + \frac{4S^3}{3L^6}\left[3\Omega^2\left(\int_0^L \frac{ds}{\rho}\right)^2 + \Omega^2 L\int_0^L \frac{ds}{\rho^2} - 4\Omega L^2\int_0^L \frac{ds}{\rho} + L^4\right] \tag{5.16}$$

Let us estimate the gain obtained by the optimization. To do this, we construct the solution of the torsion problem for a bar with a constant thickness distribution h along the contour Γ_0. Without presenting the corresponding computations, which are mainly analogous to those described above, we write down the expression for the difference $\Delta K = K - K'$ between the stiffnesses of an optimal bar and a constant-thickness bar

$$\Delta K = \frac{S^3\Omega^2}{L^6}\left[L\int_0^L \frac{ds}{\rho^2} - \left(\int_0^L \frac{ds}{\rho}\right)^2\right]$$

Applying the Cauchy-Buniakowski inequality to this expression, we conclude that $\Delta K \geqslant 0$. As is easy to note, the equality $\Delta K = 0$ is realized for a circular contour. In this case the optimal thickness distribution is constant.

The solution (5.13) — (5.15) has been found under the assumption (5.1). We investigate another case when $\min_s \rho(s) \sim H$, i.e. the case of sections of large curvature being present on the contour Γ_0 Let us again examine the optimization problem in the variables (5.2) with the sole difference that now $\rho = H\rho'$. The main relationships of the problem are obtained from (5.3) by replacing the expression ε / ρ by $1/\rho$ in (5.3). We use the small parameter method and seek the solution in the form (5.4). We limit ourselves to determining the zero approximations, which satisfy the following system of relationships:

$$\left[\left(1+\frac{t}{\rho}\right)\varphi_t^\circ\right]_t = 0, \quad \varphi^\circ(s, \ 0) = C^\circ, \quad \varphi^\circ(s, h^\circ) = 0$$

$$\varphi_t^\circ(s, h^\circ) = -\lambda^\circ, \quad \int_0^1 \varphi_t^\circ(s, 0)\, ds = -2\Omega, \quad \int_0^1 \left(h^\circ + \frac{(h^\circ)^2}{2\rho}\right) ds = S$$

Solving these relationships, we obtain the following expressions for the required quantities

$$\varphi^\circ(s, t) = 2\Omega\left[\int_0^1 \frac{ds}{\rho \ln(1 + h^0/\rho)}\right]^{-1}\left(1 - \frac{\ln(1 + t/\rho)}{\ln(1 + h_0/\rho)}\right)$$

$$\rho\left(1+\frac{h^\circ}{\rho}\right)\ln\left(1+\frac{h^\circ}{\rho}\right) = \text{const}$$

It is seen from the second formula in (5.17) that the thickness of an optimal bar is variable even in the zero approximation. If the curvature $1/\rho$ grows as s $(0 < s < 1)$ increases, then according to (5.17) the function $h^0(s)$ will decrease. It is also seen from (5.17) that the thickness distribution of an optimal bar on sections with slight curvature is constant to a sufficient degree of accuracy. This agrees with the results obtained above.

The optimal thickness distribution can similarly be investigated as a function of the curvature of the contour Γ_0 on sections for which $\rho \sim \varepsilon^m H$ $(m > 1)$. Without presenting the computations, which are similar to those presented above, we indicate the final result. We have the following asymptotic representation $\rho = h^\circ \exp(-\gamma / h^\circ)$ for the required dependence, where γ is an arbitrary constant.

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ON DEFORMATION OF CONTINUOUS MEDIA IN A WEDGE-LIKE REGION WITH SMOOTH FACES

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Using a quasi-stationary formulation, we investigate a boundary value problem on deformation of isotropic rheological media in a wedge-like region. The deformation of the medium is caused by variation in the angle between the plane sides which together form a plane diffuser, and by the flow rate of mass through this diffuser. The wedge faces are assumed to be perfectly smooth. Notwithstanding the particular properties of the medium, we succeed in determining the displacement field to within a single arbitrary function independent of the polar angle, with all boundary conditions of the problem satisfied. We show that with the quasi-stationary formulation of the problem the partial derivative with respect to time is obtained in terms of the partial derivative with respect to the diffuser angle of opening and the mass flow rate, with both these quantities assumed to be variable. The formula for the partial derivative with respect to time enables us to express any kinematic characteristic (velocity, deformation, rate of deformation, etc.) in terms of the displacements. We successfully integrate the equations of equilibrium and determine the stress field to within a single arbitrary function independent of the polar angle. In this manner we reduce the solution of the boundary value problems on deformation of continuous media in a wedge-like region with smooth faces to determining the dependence of two arbitrary functions on the radius, that is, after substituting the stress and displacement fields obtained into the defining equations of the medium in question.

Some of the problems on deformation of continuous media in wedge-like regions have been solved. Thus we have the Hamel solution [1] of the problem of flow of a viscous fluid through a diffuser, and the Schield solution [2] of the process of extruding a rigid-plastic material through a wedge-like die. Several solutions of the problems on small plane deformations of a nonlinearly elastic wedge are given in the monograph [3]. Exact solutions of the problems of large deformations of an incompressible elastic wedge with arbitrary elastic potential were obtained in [4]. In addition, numerous results of investigations of deformation of continuous media in wedge-like regions appear in [5] and others.